Designing the Problem Solving Lesson as an Organization of Students' Mathematical Activities: For Developing the Grounding of Creativity

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Key words: problem solving, mathematical activity, mathematical way of thinking

1. Introduction

It has internationally begun to attract attention to Japanese mathematics lessons and lesson studies since the investigation such as TIMSS videotape study (*e.g.* Stigler, et al., 1999). It can be said that "The Teaching Gap" (Stigler & Hiebert, 1999) is the most remarkable literature that illustrates this. However, when it is seen from the viewpoint of Japan, it is not necessarily the one that working of the teachers, the educators, and the researchers in Japan is reflected. Although it introduces a superficial style of Japanese lesson, there are few descriptions of what effort Japanese teachers and researchers are doing to make sense such a lesson. When we see an educational practice of another country, the difference with the home country will be more or less noticed. However, perhaps each effort is sure to exist in any country behind the being superficially observed.

In this paper, a practical approach concerning the mathematics lessons in Japan is proposed. These are based on what many researchers and teachers made up together. However, the mathematics lessons cannot be unique anywhere in Japan. The purpose of this paper is to propose a practical framework toward the mathematical lesson and the lesson study in the delimitation of the region, the school and the project which the author has taken part in.

This paper shows that the feature of Japanese lessons is 'problem solving'. While many researches focuses on individual learner's problem solving process, this paper does on the 'problem solving *lesson*'. That is, on the basis of many precedent researches, this paper intends to respond squarely an educational demand how to realize such a lesson.

This paper is composed as follows. First, the cultivation of the grounding of creativity of children as a fundamental viewpoint for mathematics education in this paper is discussed. The idea of 'creative practices' is shown there. Secondly, the connection between mathematics learning and problem solving is discussed. Here, students' problem solving in the lesson is characterized as an organization of their 'mathematical activities'. Furthermore, a fundamental framework of the problem solving lesson characterized like that is shown, and the discussion about

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the nature of the constitution of the lesson is developed. Finally, a brief suggestion to the lesson studies is done.

2. The grounding of creativity in mathematics education

2.1 'creation' versus 'creative'

A word 'creativity' is used in various fields and contexts. When we think this word in the field of mathematics education, in general, 'creativity' is considered as three aspects: 'originality', 'fluency', and 'flexibility' (Haylock, 1987). 'Originality' means the creation of the infrequent idea that nobody thinks till now. 'Fluency' means to create more for a matter concerned. 'Flexibility' means to create anything which is new by connecting the thing unrelated to the matter concerned at first. Indeed, these three aspects explain and describe the '*creation*' which we think now distinctly so that it has been argued in many precedent researches. However, it is not enough for us, *educators*, even if we were able to analyze no matter how detailed that a child showed such and such a '*creation*' (*we are not pure psychologists*). Rather, it will be left as an issue how children *can* realize 'creativity'. Then, how *should* we realize 'creativity' as above three aspects to children? We may deal with 'flexibility' as a problem of teaching and learning as we mention later. On the other hand, even if the value of 'fluency' can be admitted, then, how could it be a object of teaching and learning actually? In a strict meaning, there is no way that we possess now more than it is waited that an individual child completes so that it is symbolized in a kind of support like "*Do you have another idea?*" It seems to relate into another aspects of creativity, 'originality'. In actual teaching and learning process, we teachers suggest children commonly to change of a concrete viewpoint and to use the new (mathematical) tool for having so-called "other solution". Of course we do not intend to deny this. So to say, seeing from a aspect of 'originality', we want to show that children do not *create* it anymore. As shown in this point, the 'creativity' that we aim at in mathematics teaching and learning is *the creativity expected by a teacher* rather than, so to speak, *the creativity opened up* in a pure meaning of the word. In other words, we do not see the word 'creativity' as a purpose (*creation*), rather we should realize it as a method (*creative*) to a child in education. The 'creation' as purpose is impossible for stretching existing knowledge or concept. It doesn't become possible until constructing a thing newly unlike these. Such will not be really suitable to demand that for children who are the subjects of learning. So when we realize the educational aim concerned, we expect that a child grows up as ***'a person who have the responsibility for truth'*** (Balacheff, 1999), through organizing the teaching and learning environment as if each child finds, constructs, and reasons mathematical knowledge and concept by him/her-self.

2.2 Children's creative practice

Therefore, we do not aim at the 'creativity' as the pure meaning of a word in education. So we insist on '*the grounding of creativity*' in this point, and make a proposal of the following *'creative practices'* in which children are expected:

- ❖ to confront difficulty daringly and to try overcoming it;

- ❖ to manage (to practice) their mathematical knowledge or concept, etc., more than they learned;
- ❖ to be able to produce a new knowledge or concept, etc., based on what they learned.

These are also derived from the fundamental idea of the learning of mathematics in the next section.

2.3 Problèmatique in the learning of mathematics

We hope that children form desirable human nature through daily teaching and learning. Therefore we expect that children develop higher order thinking through their mathematics learning. However, children do not learn as they *will* do ‘learning’. When a child accomplished a certain activity, it seems to us from a viewpoint of teacher that it is ‘learning’ as a result. What a child does is to faces a situation, to attain awareness of a problem, and to be going to solve it. Such a problem often occurs as the difficulty that a child confronts. When it is necessary for a child to make an effort in confrontation with difficulty, ‘learning’ is concluded. Therefore, when a child may hardly make an effort to solve a problem, we do not recognize his/her problem solving as ‘learning’ of high degree. Although it is not necessary to assume absurd difficulty, when a child must make an effort for his/her problem solving very much, we recognize it with ‘learning’ of high degree (Mizoguchi, 1995*a*). This is the ontological principle of learning like as Bachelard’s description concerning parallel lines: “Les parallèles existent après, non par *avant*, le postulat d’Euclide.” (Bachelard, 1934/1975, p.143)

Hence, the followings are raised as fundamental ***problèmatiques***:

- ***epistemological*** : what kind of difficulty should children confront;
- ***psychological*** : what kind of difficulty do children really confront;
- ***learning*** : how should children overcome such a difficulty; and
- ***teaching*** : how does a teacher support children?

3. Learning mathematics and problem solving

3.1 Mathematical way of thinking and problem solving

In Japan, the Course of Study (by Ministry of Education, Culture, Sports, Science and Technology) has revised several times. The description has changed each time in relation to the goal of “Mathematics” as a subject. Each was characteristic and always led our educational practice. However, we can see ‘*development of the mathematical way of thinking*’ as a purpose of mathematics education that flows incessantly and consistently there. By the way, mathematical way of thinking is “*a way of thinking*” literally, and it is unobservable for us directly. Then, according to the foregoing section, we lay mathematical problem solving as a situation in which children find, construct, and apply their mathematical way of thinking. In other words, through their activities in problem solving, we let children’s mathematical way of thinking become evident for intending to be able to observe. This is inseparable from an issue of evaluation. Since we adopt mathematical way of thinking as our educational purpose, we have to evaluate it. As mentioned above, we are going to evaluate the primarily unobservable object through children’s activities in problem solving, which are observable. Therefore, it is children's activities in problem solving that are considered to be the next question, and we define such activities as "*mathematical activities*" (Mizoguchi, 2000).

3.2 Mathematical activity and problem solving

Hence, children's activities as objects towards / directed to our educational purposes have to be admitted that they possess mathematical values from eyes of teachers. That is, it is not enough for children to operate or to muse over anything merely. "Mathematical activities" are the *theory-laden* objects in this sense (Hanson, 1958). And it is not only the teacher side throughout but we want children the next to do it consciously. In other words, "mathematical activity" is defined that the activity which a student performs suitably in his/her mathematical problem solving and is considered the certain mathematical value to be laden. Therefore, "mathematical activity" must be placed as not an *anticipated* object so that a child does it so but a very *expected* one.

3.3 Mathematical activity and children's creative practice

In actual mathematics learning, a "mathematical activity" should be identified as unique to the individual teaching materials. Then, such a "mathematical activity" should not be specified as a sole. We have to consider diverse modes of activities as a series if we expect realization of such unique value in children's learning. In other words, when students confront a problem situation as their difficulties, they may not always reach a solution at a bound or in the unique direction. Therefore it is necessary for us to identify as a didactical issue how children accomplish true problem solving via experiencing any activity, what kind of mathematical value these activities should have, and how children evaluate such value by themselves. Hence, it seems effective to make a viewpoint to identify these activities in our lesson design. Then, the modes of children's creative practices which are mentioned above will offer one of framework of viewpoint.

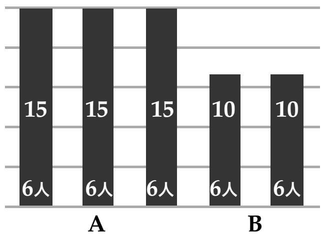
For instance, let us think about the following example. The sixth grade students are posed the following problems in a situation of "the use of mean". The problem is that when two groups A and B collected empty cans as a list, find the average number of can per one person of the whole group. Then, expected mathematical activities in this case are as follows.

Group	The number of people	The average number of cans per one person
A	18	15
B	12	10

Activity A: A student don't have any perspective for solving this problem at the beginning. Then, a teacher distributes a table which shows the original data of two groups. The table is made to show *six data at one line* intentionally. The student considers two groups to be one whole, and calculates the average number by (total number of empty cans) ÷ (total number of people), which the student have already learned. This is considered as the mode '*to confront difficulty daringly and to try overcoming it*'. In fact, it is quite toilsome for the student to calculate the sum total of 30 data even if using a calculator. However, to calculate mean in this method is the most in principle. If a student is going to accomplish it daringly because this is having learned already, it should be praised as really splendid activity.

Activity B: However, on the other hand, we also expect that a student thinks that the method mentioned above is not so good and how to do it more skillfully. Then, if a student can calculate the sum total of each group by the original list, the activity A would be refined more. This is considered as the mode '*to manage (to practice) their mathematical knowledge or concept, etc., more than they learned*'. Expressing original data by a graph, it will be seen a uneven state. However, the list shown at first means to make even the graph every two groups. If a student can see so, it is the overt that he/she understands a sense of mean better.

Activity C: Furthermore, based on *Activity B*, a student could see as follows. When it sees the number of people shown on the original list, they are six multiples both.[1] By paying attention to this point, based on the nature of mean, that is "to make even", a student can find the demanded mean more skillfully than activity B. And it should notice that a student creates a new way of seeing data. So it is considered as the mode '*to be able to produce a new knowledge or concept, etc., based on what they learned*' in this sense. In fact, the table distributed to necessary students in activity A was prepared as expected such *Activity C* over there.



4. Didactics of mathematics

4.1 Teaching materials [2] study

Though it is pointed out in all subjects equally, the most important work for didactics is teaching materials study. This has been also emphasized untill now. The problem that we consider here is how to conduct teaching materials study in didactics of mathematics for '*developing the grounding of creativity*' or '*developing children's creative practice*'. As we mentioned above (*cf.* 1.3), since we consider the learning of mathematics basically so that a student confronts a problem as his/her difficulty, then we concern such didactics of mathematics as what kind of difficulty a student should confront; what kind of difficulty he/she really confronts; how a student should overcome such a difficulty; and how a teacher supports him/her, it is considered at least

[1] In the graph, "人" means persons in Japanese.

[2] Teaching Material means here that they consist of mathematical contents as resources, and problems for the teaching and learning process.

to execute teaching materials study about two following phases.

Phase 1: Study from a standpoint of mathematics

The first is study from a mathematical viewpoint. This does not mean merely that we should know a mathematical background as a scientific discipline of contents formed as the teaching materials. Of course it is an important study. However, it is the core work to analysis mathematical value for forming the content as the teaching material, its connection with other contents, and its difficulty.

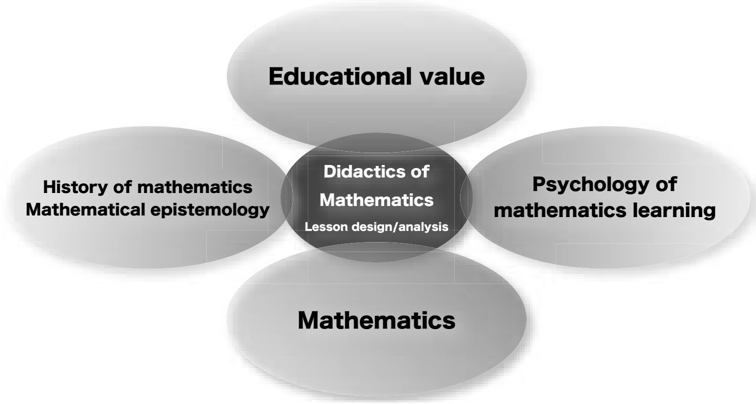


Fig. 1 Didactics of Mathematics

For example, in teaching of "measurement of angle" in the fourth grade in Japan, it could be the study from a standpoint of mathematics to obtain the following knowledge.

> An angle is one of the most basic component of the geometric figure. At the same time, an angle is the most basic object of the measurement, that is *quantity*, like as length. In the first situation children should think why do we measure it? By setting it the numerical value –or evaluating– through the measurement, it is the aim to convert the measured value into an object of a computation, that is *number*. How do students accomplish such evaluating? Students already experience evaluating quantities of length, bulk, weight, and etc., so far. That is, students experience measuring some quantities by deciding a base unit of each one, and evaluating the object equivalent to how many times of the unit. However, it is needed as a premise to recognize whether such quantity is possible to measure in the first situation. An angle will be the very object that it should be confirmed the need for having such recognition. For this, what kinds of activities should we expect to children? Before recognizing an angle as for quantity, namely an object of the measurement, a student passes through activity of comparison. This is not only a reason for starting merely from a light cognitive burden psychologically. The quantity doesn't exist as an quantity from the beginning for students, and it is an extremely important activity that a student recognize it as quantity. Even if this is never especially noted for length and weight, a student was able to recognize it, so to speak, intuitively. This in itself is not a denied fact. However, it is not necessary so for the bulk, and also for the angle. For this, expressing by

the measurement numerically is done attempting the achievement of the intention to try to manage to universalize the bigness and smallness by starting from first of all distinguish at least big and small possible. This is a basic way of thinking of objectification as quantity by comparison. Then, in the case of an angle in particular, how should this be done? The teaching of the size of quantity includes to enrich "the quantity sense" besides the measurement. In other words, it is teaching of a sense for / about / of quantities. Regarding to an angle, the quantity sense with a dynamic operation of rotation is requested unlike other quantities already learned. It is because the comparison of the size of angle is actually impossible if this operation is not accompanied. It means that uniqueness should be guaranteed that big and small being decided by the comparison. For instance, a dynamic operation of rotation is indispensable to distinguish 30° from 150° or 45° from 315°. This also becomes a basic premise at the same time when *measuring*. That is, the angle has periodicity while the measured values were monotone increasing in the measurement of already learned quantities. For this, if the form preservation with other quantities is intended, the dynamic nature of quantity is requested for an angle against the static nature of quantity for others.

Phase 2: Study from a standpoint of problem solving

The second phases of the teaching materials study is from the standpoint of problem solving. This is to design what problems and their sequences (the plan of teaching unit) should be prepared concretely, and what kinds of activities could be whether students appreciate such mathematical value and overcome difficulties, when students learn the content analyzed by the study from the first phase, that is, a mathematical standpoint (to transpose it as a teaching material).

For instance, obtaining the following knowledge about "Addition with carrying" (the first lesson of the unit) in the first grade could be the study from this phase.

It can be said that the addition with carrying is an appropriate teaching material to construct new mathematical knowledge based on already learned one for the first grade students. In the textbook, "8+3" is often selected as a numerical value usually, even if there are somewhat of differences of the problem situation. However, will the problem solving to expect the achievement of mathematical value of "Addition with carrying" in this numeric setting be possible? Or, might a student be able to experience the difficulty that should be overcome included in there potentially? The most important point in this learning situation is to construct the operation carrying based on the complement in terms of ten when the sum is more than ten. It is necessary for this point to be organized as expected mathematical activities in students' problem solving. Therefore, in the lesson, we set the situation where 8+6 are asked –a concrete problem is omitted here– and want to expect students' evolution of the following mathematical activities.

Activity A: Addition by counting

Teacher's support: "Let's think about the operation that can be

○	○	○	○	○	○	○	○	●	●	●	●	●	●
1	2	3	4	5	6	7	8	9	10	11	12	13	14

shown in a expression."

Activity B: The complement in terms of ten

B1: ○○○○○　○○○
●●●●●　●　☜ It is necessary to decompose both.

B2: ○○○○　○○○○○●
●●●●●　☜ It is necessary to add a big number comparatively though it is an excellent method of knowing how many it is necessary to make ten.

B3: ○○○○○○○○●●
●●●●　☜ More excellent method.

Activity C: A student can represent the operations of *Activity* B in each expressions, and explain by using it.

4.2 The lesson of mathematical problem solving

On the basis of teaching materials study we design a lesson as teaching and learning process, which is understood as an interaction by teacher, student, and teaching material (Fig. 2), especially in mathematics, it is taken as the mathematical problem solving. As we mentioned above in 1.3, it is the student's learning to solve/overcome a confronted problem/difficulty, and a teacher may be set in the environment so that students accomplish their learning successfully.

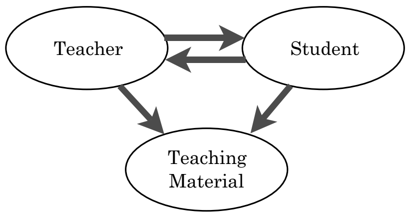


Fig. 2 Didactical Triangle

In general, as for the mathematical problem solving *as a form of lesson*, a basic flow shown in Fig. 3 has been presented in a precedent studies (e.g. Ito, 1993). In addition, we undertake further enhancement of the lesson as a mathematical problem solving by organising expected mathematical activities (Fig. 4). In the following, we argue along this fundamental framework.

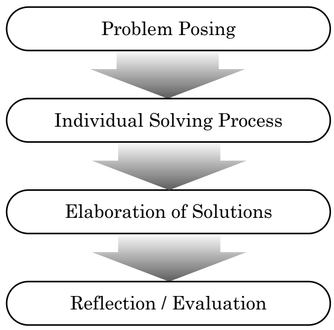


Fig. 3 Basic Flow of Lesson as Problem Solving

4.2.1 Posing a problem: Setting a "good problem" and importance of estimation

As mentioned above, a teacher analyzes in teaching materials study (from a standpoint of mathematics) what kinds of difficulties students should confront. Next, a teacher needs to study how students really confront and are expected to have a perspective for solving a problem (from

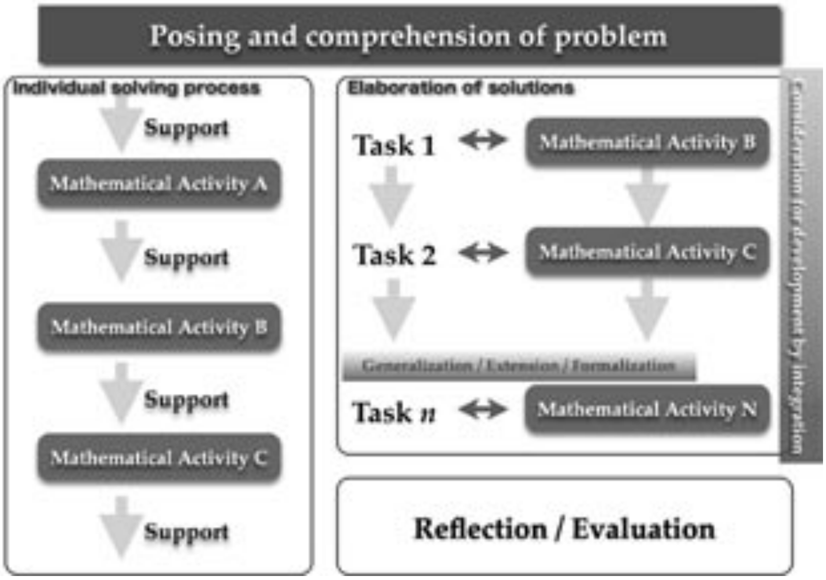


Fig. 4 A Model of Lesson as an Organization of Mathematical Activities

the standpoint of problem solving). In the situation of problem posing, then, it is important that a devised "good problem" enables students to have the awareness of the problem truly as their own, not enough in posing of the given problem situation to them merely. At this time, to promote the problem solving ability of a individual student, it should avoid to confirm an excessive perspective deciding its solution in the whole classroom. A necessary thing is analysis of a given problem situation, and comprehension of the true subject in this problem *(problem formulation)*. Then, it will be effective in many cases to estimate the solution of problem. For instance, in the learning of the addition and subtraction of fractions with the different denominators, that is equivalent fraction, in the sixth grade, it would be assumed that the following problem situation (a concrete problem is omitted here) were posed.

《Ordering of $\frac{2}{4}(\ell)$, $\frac{2}{3}(\ell)$, $\frac{3}{4}(\ell)$》

The estimation expected here is that comparing $\frac{2}{4}$ and $\frac{2}{3}$. The latter is bigger than the former because numerators are the same and denominators are different (the former is two of four equally divided parts and the latter is two of three equally divided parts). Comparing $\frac{2}{4}$ and $\frac{3}{4}$, the latter is bigger than the former because denominators are the same and numerators are different. Therefore the true problem that should be solved is which is large in $\frac{2}{3}$ and $\frac{3}{4}$, and moreover *how much one from the other*. Such an activity is needed on analysis of a problem. In this case, from the estimation done here, it is possible to obtain a perspective that is able to order two fractions by generating the common denominator (or numerator).

4.2.2 Individual solving process: Expected mathematical activities and teacher's supports to evolve them

It is usually seen that a teacher anticipates students' reactions in the lesson. It is not denied itself, rather is desirable for the teacher to be able to anticipate about what reaction of each student if the teacher is in charge of the class. However, the following indication about 'instruction according to the individual differences' are extremely important for designing the lesson: "It doesn't become the problem immediately that two learners A and B executing different solutions (ideas). For A and B, it is asked whether a teacher needs the same support or the different, and, as a result, it is decided how many supports are needed." In other words, there is a reason of instruction according to the individual differences (anticipation of solutions) not in possible patterns of learners' solutions but in a demand of how many kinds a teacher should support in the lesson. Then, what such supports should be set to, and why. We would like to focus on students' mathematical activities. 'Mathematical activity' mentioned here doesn't only mean an operative activity or an experimental activity alone. As we mentioned above (*cf*. 2.2), 'mathematical activity' is defined as the activity that a student performs appropriately in his/her mathematical problem solving and is considered the certain mathematical value to be laden, and, it is a teacher that sees the student's activity as mathematical value laden. A teacher aims at such laden mathematical value devolving on a student as a result of learning. Therefore, the student's behavior or thinking which is considered as an 'anticipated reaction' so far is regarded as an 'expected activity (by a teacher)', and then it will be formulated what mathematical value is laden on each activity. At this time, it is not that the previous "anticipated reaction" is not to play any role at all. Even if no matter how splendid mathematical value will be laden, we cannot expect activity far apart from the real state of a student. Moreover, "expected" means to include the student's reaction not anticipated necessarily, then, "support" is considered as the next didactical problem.

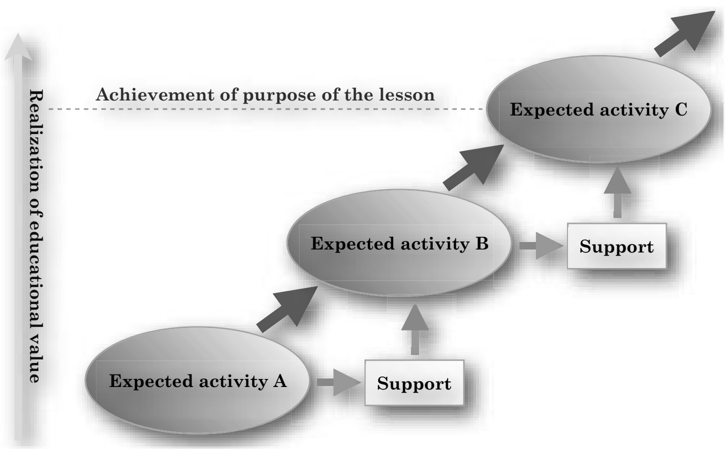


Fig. 5 A Model of Evolution of Mathematical Activities

Support is a way according to student's individual difference. However, as what should be noted, support is not like only leading the direct solution of a problem at once. As mentioned above, we think of the expected mathematical activities. As for them, each is of value laden. Therefore, a student may not only experience any one activity. That is, it is worthwhile that all students experience these activities through the whole lesson (not only at the situation of individual solution). Therefore, support must be the one raising the level of student's activity adequately as a mediation of each activity. Going through the process of such individual solution, each student could experience the situation of elaborating these solutions truly in the next.

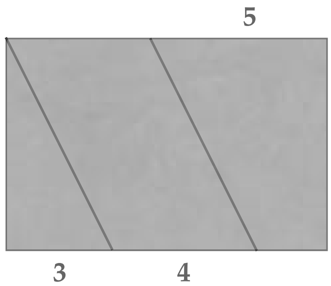
4.2.3 Elaboration of solutions: Consideration for development by integration

As confirming at first, the situation of elaboration is neither *a recital* of students' solutions nor *un concours* . Elaboration of solutions is done to intend a social construction of mathematical knowledge and concept. We see the student's learning as overcoming difficulty in the above. Although a student should overcome such a difficulty individually, the knowing (knowledge, concept, etc.) achieved by overcoming should not be personal (*the discussion between 'connaissance' and 'savoir'*, Balacheff, 1990). That is, a teacher aims at students attaining socially shared knowledge/concept, and therefore sets the situation of individual solution for evolution of each student's personal conception as the didactical problem (Balacheff, 1990, 1991; Mizoguchi, 1995*b*). In this sense, it is for elaboration that individual solving process is, not elaboration for individual solving process. In other words, elaboration does not mean like that a teacher want to introduce any student's solving to others since he/she did well in the process of individual solving. We prepare the process of individual solving to elaborate successfully at first, because we couldn't keep away from individual differences of students by any means, and it is impossible to enter the situation of elaboration at a beginning of a lesson usually. Therefore, it becomes a required matter that all students can participate in the elaborating process. In Fig. 4, the elaboration of solutions is started at *Activity B as one case*. In such a case, it is necessary in the individual solving process that all students have already been achieving *Activity A* , and *Activity B* was achieved or engaged in. Therefore, *Activity B* plays a role as a '*ticket for participation*' at the beginning of the elaboration. At this time, 'Task' posed for taking up the activity in the elaboration may be the same as the corresponding 'Support' in the the individual solving process basically. It depends on the following reasons. Each 'Support' intends to promote a student's activity, and therefore, each one has an important meaning for the student. As for a student who was able to finally start *Activity B* in the individual solving process, it is necessary to begin the collaborative solving activity –*elaboration*– in the whole classroom from there because the student does not yet experience 'Supports' of afterward. As for a student who managed *Activity B* and supported afterward by the teacher in the individual solving process, the student also begin the cooperative solving activity at this point because the student has not achieved the activity expected by the 'Support'. As for a student who engaged in or accomplished *Activity C* , there are two possibilities. The first is a student who has evolved his/her activities according to the sequence of the teacher designed. The student is offered a precious seat for looking back on his/her own activities. The second is a student who achieved *Activity C* for oneself at the beginning of the individual solving process. Of course, though it is needless to say that a teacher supports similarly in the situation of solving individually, in many cases, such a student may not understand necessarily the true value of his/her activity accomplished. Therefore, the student could evaluate truly each value and evolution of a sequence of his/her activities through experiencing the elaboration process based on 'Tasks' by a teacher.

What discussed above is, if anything, concerning a matter of the form of lesson. It is necessary for us to answer a fundamental problem what "elaboration" should be, as a more important thing. It is not enough only in obtaining the solution of the given problem by the lesson of a mathematical problem solving. A teacher aim at students constructing mathematical concept, knowledge, and skill, etc., through such a problem solving process. Therefore, it could be considered that to obtain the solution of a problem is the beginning of true elaboration. It is for this that locates *Activity N* following *Activity C* in Fig. 4. Again, as reflecting the purpose of mathematics education, it is '*fostering of the mathematical way of thinking*', and we reduce it to promote student's problem solving ability. It is also said that '*fostering of the mathematical way of thinking*' means that students can do the creative activity worthy of mathematics (Nakajima, 1981). The '*consideration for development by integration*' is indicated as a typical creative activity. 'Generalization', 'extension', and 'formalization' are shown in Fig. 4 as actual modes of such '*consideration for development by integration*'. That is, it is not expected achieved to construct mathematical knowledge and concept through problem solving until by these considerations. In this sense, so-called "development" should be considered not only in the end of the teaching unit but every lesson (especially, in the elaboration). In the following, we discuss 'generalization' and 'extension' in particular.

'Generalization' and 'Extension'

For instance, let us think the following problem solving process. In the right figure, find the ratio of areas: a triangle, a parallelogram, a trapezoid (the sixth grade level in Japan). Expected mathematical activities to this problem are as follows. (The ratio of unknown sides was found through analysis of the problem.)



Activity A: *Applying some concrete numerical value to "height" and calculating an area of each figure.*

In activity A, height is installed virtually because necessary height for demanding the area is not given, and then the ratio of the areas is possible to be calculated. Even if height is any number then, knowing of the ratio of areas being invariable is demanded for a student.

Activity B: *Each figure is divided into the triangle, and the ratio of the original areas is reduced to the ratio of the sum of the bases.*

In activity B, it is achieved to solve the concerned ratio concisely by using only one mensuration formula, while the formula of each figure was used in activity A.

Activity C: *Each figure is considered to be a trapezoid, and it reduces the ratio of the original areas to the ratio of (upper base + lower base) of each.*

Although founding the solution of problem by one viewpoint in activity B, the operation of adding supplementary lines were needed for that. In activity C, the problem is solved by not needing such an operation, rather making a change to the viewpoint itself. It may be described more in detail as follows. In the use of a mensuration formula in *Activity B*, it divided the figures into individual triangles to be able to apply the formula directly. On the other hand, it is not to make an operation on the figures to be easy to use the mensuration formula of trapezoid in *Activity C*. Based on a definition of a trapezoid, it is able to apply this formula to a wider range by seeing a parallelogram as a trapezoid (the inclusion relationships of quadrilateral). Furthermore, it widens such a way of seeing in a (right) triangle (upper base is zero). The mensuration formula of trapezoid was not made any change at all. So to speak, it could be pointed out to enlarge the coverage for a known mathematical idea more than before. In other words, 'generalization' of the use of a mensuration formula of trapezoid was done. At this time, we confirm the followings about 'generalization'. That is, what the mathematical idea stored in a process of 'generalization' is, and what is *particular* so far is able to be integrated by such a mathematical idea.

Comparing with 'generalization', we will examine a situation of multiplication of the decimal that is the typical case as 'extension' in mathematics learning. Whereas a meaning of multiplication of whole numbers is "additive repetition", since it cannot explain the situation of multiplication of decimals well, it is to determine the new meaning of multiplication as "proportional reasoning", that is, $A \times p$ is meant as "the size to p when seeing A to 1".

At this time, a new meaning of the multiplication of decimals is not made based on the meaning of whole numbes. Rather, after a new meaning was made, it would be integrated into the new through comparison with the meaning of whole numbes. That is 'extension'. Extension is defined as follows generally: "Meaning M stands up in domain D. If meaning M' which stands up in wider domain D' including domain D is equivalent to meaning M when limiting D' to D, M' is the extension of M". Hence, to extend a meaning of multiplication of decimals, the following modes of activities will be demanded at least: 1) knowing that a meaning of multiplication stood up in the case of whole numbers is inconvenient in the case of decimals; 2) constructing a meaning to stand up in the case of decimals (at this point, it cannot be declared yet rigorously that it is "*multiplication*"); 3) comparison between a current meaning and a new constructed one; 4) integrating a current meaning into a new one.

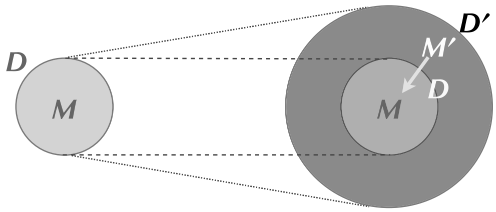
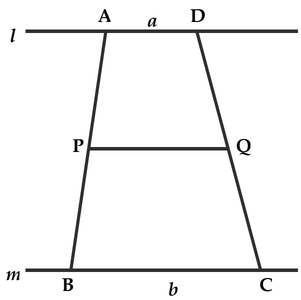


Fig. 6 A Model of Extension

What we want to consider here is a difference between 'extension' and 'generalization'. As we mentioned above, 'extension' is not possible until a meaning (or concept) to be extended was given. While that, 'generalization' is to generalize the known literally, that is, there is knowing with a kind of constant direction. Therefore, both would take on phases considerably different from the epistemological perspective. 'Generalization' and 'extension' must be need and important as for developing the mathematical way of thinking. Although both might often be confused, it should be understood that they are distinguished definitely based on above discussion. Then, different approach would be demanded of the lesson (in particular, the process of elaboration).

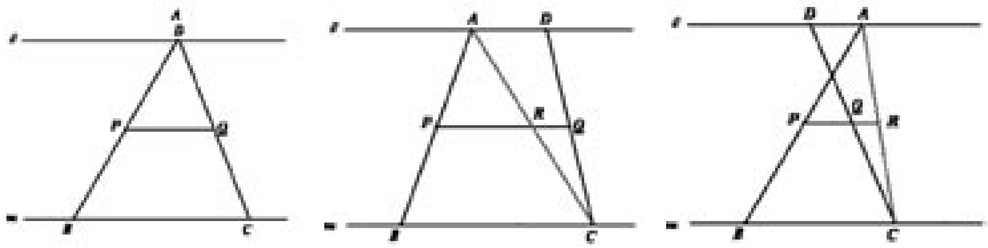
A supplementary example

Problem: As a given figure, if $l \mathbin{/\!/} m$, P, Q being the midpoint of AB, CD respectively, and $AD = a$, $BC = b$, then find the length of PQ by a, b.



What is really asked in this problem is "how the length of PQ changes by the position of point A, B, C, and D".

The case that point A and D coincide would be the most easy to understand. In this case, $PQ = \frac{b}{2}$ is deduced directly by Two Midpoints Theorem (learning at the eighth grade, in Japan). In the case of the original figure of the problem, $PQ = \frac{a+b}{2}$ is deduced by applying the first case twice. Then, how does it occur if CD intersects AB in the inside of the parallel lines? Generalizing the method in case of the second, $PQ = \frac{b-a}{2}$ is deduced by applying Two Midpoints Theorem twice, after all.

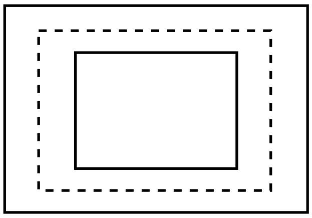


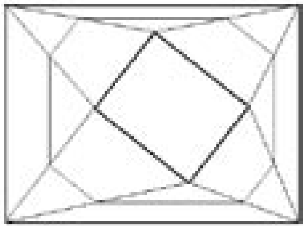
However, this way of thinking depends on intuition to figure very much. That is, it is necessary to explain why the numerator is $(b - a)$ in the third case while $(a + b)$ in the second. In the learning of the 8th graders, it might be possible to decide positive and negative length by making some conditions to the length. More, in high school mathematics, using the idea of the vector, seeing as $|\overrightarrow{AD}| = |a|$ and $|\overrightarrow{BC}| = |b|$, it is expressed without depending on the position of four points A, B, C, and D, such as;

$$PQ = \frac{\left|\overrightarrow{AD} + \overrightarrow{BC}\right|}{2} = \frac{|a+b|}{2} \qquad \text{(both } a \text{ and } b \text{ might be negative values).}$$

This would be the 'extension' of Two Midpoints Theorem, that is, it is possible to integrate every case by such a viewpoint. Therefore, we can see that a consideration for development is done by such integration.

On the other hand, it can be considered that the length of PQ is a mean of AD and BC if this situation is seen in elementary school mathematics. In other words, it can be seen that a mathematical idea '*mean*' is abstracted and refined. A similar activity in the elementary school is in the following situation. The problem is to find the length of "way" (a dotted line part of the figure) made in the middle between the outside rectangle and the inside one (it is just located in "center") as shown in the figure. Now, the length around an outside rectangle being L, the length around an inside rectangle being ℓ , the length to find is obtained as $\frac{L+\ell}{2}$. This is from the idea of "mean". Then, how can you see when an inside rectangular position is not "center"? Translating an inside rectangle, it is possible to process it by expanding the original idea ('generalization'). When rotating, how is it possible to process it? Only a conclusion here, in defining "a mid way" as shown in the right figure, a current form (the above expression) can be preserved (using Two Midpoints Theorem). Moreover, because of including the case of translation, it is possible to see this way of thinking to be integration by 'extension'.





Connecting mathematical activities

In above discussion, although we describe the consideration for development by integration by means of 'generalization' or 'extension' as the example, we are going to confirm how this is realized in actual lesson practice. Of course, whether 'generalization' or 'extension' is decided from how to teach a concerned mathematical content through the teaching materials study. However, students doesn't think those intentionally at the beginning while the teacher at least is aiming at. Here, a teacher doesn't say justly "let's generalize (or extend)!" In any approach, it could be by connecting individual mathematical activities to promote the consideration for development by integration. That is, connecting between the activities is a start line of 'generalization' or 'extension' according to a concerned teaching material. Hence, it may be said that the teacher's first target in the process of elaboration is how individual mathematical activities is connected.

4.2.4 Reflection / Evaluation: New problem to the next lesson

The last phase in the lesson of the problem solving is to look back on or to solve the evaluation problem for confirming the achievement of the current learning. An evaluation problem is far more difficult than the problem at the current lesson is often posed. The implication is to confirm mathematical knowledge or concept, etc. constructed through the problem solving at the lesson. Setting the evaluation problem is desirable for students to be surely conscious of what they learned in that lesson, without being too complicated. Moreover, it could be an important viewpoint of designing the lesson so that students have a perspective of the next lesson by looking back on the current learning through solving such an evaluation problem. This is related to plan the teaching unit.

5. Planning the teaching unit

We can see the following citation in relation with the consideration for development by integration; “Although ‘integration’ is aiming that students themselves can think creatively from such a viewpoint, to the base, first of all, the teacher should comprehend the connection of contents systematically, and pose a problem to students on the standpoint of integration / development. ... In fact, however, it is not rare that there are problems rather in such respect.” (Nakajima, 1981; *translated by the author*) Lesson of mathematical problem solving should be insufficient only to be evaluated every hour. It is necessary to plan the problem solving through the unit at least. The development of the set of problems through the unit has been pointed out so far. Here, to enhance this suggestion further more, we would like to design the planning of the teaching unit shown in Table 1. While ‘Purpose of lesson’ means the target for students, ‘The core idea’ means to design the teaching approach to the target. In other words plainly, it is the focus of intention of how to consider for development by integration (construction of new mathematical knowledge and concept) in each lesson. As for actual modes, it should be expected at each lesson whether how to generalize, to extend, or to formalize (see Fig. 4).

Table. 1 the planning of the teaching unit

No.	Learning content	Purpose of lesson	The core idea	Problem	Main math activities

Though it is desirable that such a plan of the teaching unit is designed, it will be anticipated that the realization is very difficult by some factors like physical and time. Nevertheless it is expected that such a plan designed is shared as fortune, when the meeting of the lesson study is carried out, or by a team of teachers cooperatively. In addition, it will be an important resource to evaluate a lesson scientifically. That is, it is a resource as seeing what and how the lesson have progressed in comparison with last year, 5, or 10 years ago.

6. Cooperative lesson study

I want to point out the following respect at the end of this paper. Lesson study often stands up on the effort only of the lesson teacher oneself. This is not denied, however, it isn't decided the lesson teacher easily in the meeting, moreover his/her loads tend to become extremely large. From that, it is hard to be a fruitful discussion in the meeting, on the contrary, it tends to become a cause that dissatisfaction remains in the lesson teacher who had a hard time. In this sense, it could be suggested for shifting the situation of the meeting from "looking at the lesson mutually" to "designing and talking about the lesson together". That is, it is necessary to see the lesson design in, at least, the lesson study meeting as the cooperative team work to discuss productively. As a strategy for this, it would be proposed to observe and record the extracted students so that we can see how the activity of such a student changed or not by the teacher's support. Such discussion based on the record will be productive.

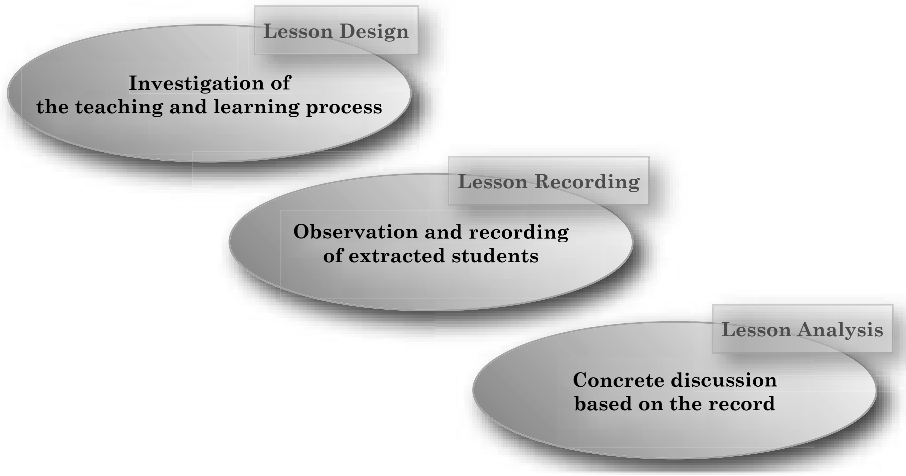


Fig. 7 A Model of Lesson Study

7. Final Remarks

Many of researchers in Japan, also the author, value relations with the school and school teachers. It means that a researcher is concerned as an advisory cooperator to such a school that becomes the subject of the research, not researcher doing participant / non-participant observation merely. As for the author, I visit to schools for about 80 days in the year. The researches related to the lesson in Japan have been built from the cooperative approaches and mutual trusts between schools / school teachers and researchers.

I would like to discuss with another paper how to design and analysis of the lesson actually by such a cooperative approach.

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